Final Exam
Philosophy 470

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Problem 1. Recall the language of arithmetic L_{Ar} with symbols:

- Constant Symbols: $\underline{0}$
- Predicate Symbols: only equality $\doteq$
- Function Symbols: $\underline{S}$

Furthermore, recall the intended L_{Ar}-structure of arithmetic $\mathcal{N}$:

- $|\mathcal{N}| = \mathbb{N}$
- $\underline{0}^{\mathcal{N}} = 0$
- $\underline{S}^{\mathcal{N}}(n) = n + 1$ for all $n \in |\mathcal{N}|$

(a) Show that $\mathcal{N}$ model the sentences

$$\psi_1 := \forall x(\neg \underline{S}x \doteq \underline{0}) \tag{PA1}$$
$$\psi_2 := \forall x \forall y(\underline{S}x \doteq \underline{S}y \rightarrow x \doteq y) \tag{PA2}$$
$$\psi_3 := \forall y(\neg y \doteq \underline{0} \rightarrow \exists x(y \doteq \underline{S}x)) \tag{PA3}$$

and for any $\varphi \in F_{L_{Ar}}$ with $FV(\varphi) = \{x\}$

$$(\varphi^x_{\underline{0}} \wedge \forall x(\varphi \rightarrow \varphi^x_{\underline{S}x})) \rightarrow \forall x \varphi \tag{PA4}$$

Proof. Let us consider ψ_1. $\mathcal{N} \vDash \psi_1$ if and only if $\mathcal{N} \vDash \forall x(\neg \underline{S}x \doteq \underline{0})[\sigma]$, where σ is an arbitrary $\mathcal{N}$-assignment. This is true if and only if for any $a \in |\mathcal{N}|$, $\mathcal{N} \vDash (\neg \underline{S}x \doteq \underline{0})[\sigma(x|a)]$. This is true if and only if $\langle \overline{\sigma(x|a)}(\underline{S}x), \overline{\sigma(x|a)}(\underline{0}) \rangle \notin \doteq^{\mathcal{N}}$. This is the same as saying $\langle \underline{S}^{\mathcal{N}}(a), 0 \rangle \notin$

$\doteq^{\mathcal{N}}$. Which means $\langle (a+1), 0\rangle \notin \doteq^{\mathcal{N}}$. We interpret this in our metalanguage to mean that for any $a \in |\mathcal{N}|$ ($a \in \mathbb{N}$), $a+1 \neq 0$. We know this to be true, because the only value of a for which $a+1 = 0$ is $a = -1$. But $-1 \notin \mathbb{N}$, so $-1 \notin |\mathcal{N}|$. Thus it is true that for any $a \in |\mathcal{N}|$ ($a \in \mathbb{N}$), $a+1 \neq 0$. By our line of reasoning, this implies that $\mathcal{N} \vDash \psi_1$.

Consider ψ_2. $\mathcal{N} \vDash \psi_2$ if and only if $\mathcal{N} \vDash \forall x \forall y (\underline{S}x \doteq \underline{S}y \to x \doteq y[\sigma])$, where σ is an arbitrary $\mathcal{N}$-assignment. This is true if and only if for all $a \in |\mathcal{N}|$, for all $b \in |\mathcal{N}|$, $\mathcal{N} \vDash (\underline{S}x \doteq \underline{S}y \to x \doteq y)[\sigma(x|a)(y|b)]$. This is true if and only if $\mathcal{N} \nvDash \underline{S}x \doteq \underline{S}y[\sigma(x|a)(y|b)]$ or $\mathcal{N} \vDash x \doteq y[\sigma(x|a)(y|b)]$. We are done if $\mathcal{N} \nvDash \underline{S}x \doteq \underline{S}y[\sigma(x|a)(y|b)]$, so let us assume that $\mathcal{N} \vDash \underline{S}x \doteq \underline{S}y[\sigma(x|a)(y|b)]$.

This assumption tells us that $\langle \overline{\sigma(x|a)(y|b)}(\underline{S}x), \overline{\sigma(x|a)(y|b)}(\underline{S}y)\rangle \in \doteq^{\mathcal{N}}$. Which means $\langle (a+1), (b+1)\rangle \in \doteq^{\mathcal{N}}$. In our metalanguage, our assumption tells us that for any $a \in |\mathcal{N}|$, for any $b \in |\mathcal{N}|$ ($a, b \in \mathbb{N}$), $a+1 = b+1$. We know, in our metalanguage, that simple algebra tells us that this is equivalent to $a = b$.

We wish to show that $\mathcal{N} \vDash x \doteq y[\sigma(x|a)(y|b)]$. This is true if and only if $\langle \overline{\sigma(x|a)(y|b)}(x), \overline{\sigma(x|a)(y|b))}(y)\rangle \in \doteq^{\mathcal{N}}$. That is, $\langle a, b\rangle \in \doteq^{\mathcal{N}}$. In our metalanguage, this means, for any $a \in |\mathcal{N}|$, for any $b \in |\mathcal{N}|$ ($a, b \in \mathbb{N}$), $a = b$. This is precisely what we have from our previous assumption. Thus, it is true that $\mathcal{N} \vDash x \doteq y[\sigma(x|a)(y|b)]$. By our line of reasoning, this implies that $\mathcal{N} \vDash \psi_2$.

Consider ψ_3. $\mathcal{N} \vDash \psi_3$ if and only if $\mathcal{N} \vDash \forall y (\neg y \doteq \underline{0} \to \exists x (y \doteq \underline{S}x))[\sigma]$ where σ is an arbitrary $\mathcal{N}$-assignment. This is true if and only if for any $b \in |\mathcal{N}|$, $\mathcal{N} \vDash (\neg y \doteq \underline{0} \to \exists x (y \doteq \underline{S}x))[\sigma(y|b)]$. This is true if and only if $\mathcal{N} \nvDash \neg y \doteq \underline{0}[\sigma(y|b)]$ or $\mathcal{N} \vDash \exists x (y \doteq \underline{S}x)[\sigma(y|b)]$. We are done if $\mathcal{N} \nvDash \neg y \doteq \underline{0}[\sigma(y|b)]$, so let us assume that $\mathcal{N} \vDash \neg y \doteq \underline{0}[\sigma(y|b)]$.

This assumption tells us that $\langle \overline{\sigma(y|b)}(y), \overline{\sigma(y|b)}(\underline{0})\rangle \notin \doteq^{\mathcal{N}}$. That is, $\langle b, 0\rangle \notin \doteq^{\mathcal{N}}$. In our metalanguage, our assumption tells us that for any $b \in |\mathcal{N}|$ ($b \in \mathbb{N}$), $b \neq 0$.

We wish to show that $\mathcal{N} \vDash \exists x (y \doteq \underline{S}x)[\sigma(y|b)]$. This is true if and only if there is some $a \in |\mathcal{N}|$ such that $\mathcal{N} \vDash y \doteq \underline{S}x[\sigma(y|b)(x|a)]$. This is true if and only if $\langle \overline{\sigma(y|b)(x|a)}(y), \overline{\sigma(y|b)(x|a)}(\underline{S}x)\rangle \in \doteq^{\mathcal{N}}$. This means $\langle b, a+1\rangle \in \doteq^{\mathcal{N}}$. In our metalanguage, this means that for any $b \in |\mathcal{N}|$, there is some $a \in |\mathcal{N}|$ such that $b = a+1$. Naturally, we would take such an a to be $b-1$. However, in the case that $b = 0$, $b-1 = -1$, and this would mean that $a \notin |\mathcal{N}|$. Recall, though, that our previous assumption prevents $b = 0$, so we have eliminated the only case in which this is not true. Thus it is true that $\mathcal{N} \vDash \exists x(y \doteq \underline{S}x)[\sigma(y|b)]$. By our line of reasoning, this implies that $\mathcal{N} \vDash \psi_3$.

For any $\varphi \in F_{L_{Ar}}$ with $FV(\varphi) = \{x\}$, consider $(\varphi^x_{\underline{0}} \wedge \forall x(\varphi \to \varphi^x_{\underline{S}x})) \to \forall x\varphi$. $\mathcal{N} \vDash (\varphi^x_{\underline{0}} \wedge \forall x(\varphi \to \varphi^x_{\underline{S}x})) \to \forall x\varphi$ if and only if $\mathcal{N} \vDash (\varphi^x_{\underline{0}} \wedge \forall x(\varphi \to \varphi^x_{\underline{S}x})) \to \forall x\varphi[\sigma]$, where σ is an arbitrary $\mathcal{N}$-assignment. This is true if and only if $\mathcal{N} \nvDash (\varphi^x_{\underline{0}} \wedge \forall x(\varphi \to \varphi^x_{\underline{S}x}))[\sigma]$ or $\mathcal{N} \vDash \forall x\varphi[\sigma]$. We are done if $\mathcal{N} \nvDash (\varphi^x_{\underline{0}} \wedge \forall x(\varphi \to \varphi^x_{\underline{S}x}))[\sigma]$, so let us assume that $\mathcal{N} \vDash (\varphi^x_{\underline{0}} \wedge \forall x(\varphi \to \varphi^x_{\underline{S}x}))[\sigma]$.

This assumption tells us that $\mathcal{N} \vDash \varphi^x_{\underline{0}}[\sigma]$ and $\mathcal{N} \vDash \forall x(\varphi \to \varphi^x_{\underline{S}x})[\sigma]$. The latter means that for any $a \in |\mathcal{N}|$, $\mathcal{N} \vDash \varphi \to \varphi^x_{\underline{S}x}[\sigma(x|a)]$. Furthermore, this is true if and only if $\mathcal{N} \nvDash \varphi[\sigma(x|a)]$ or $\mathcal{N} \vDash \varphi^x_{\underline{S}x}[\sigma(x|a)]$. By the Substitution Lemma, $\mathcal{N} \vDash \varphi^x_{\underline{0}}[\sigma]$ means that $\mathcal{N} \vDash \varphi[\sigma(x|0)]$. So it cannot be the case that for any $a \in |\mathcal{N}|$, $\mathcal{N} \nvDash \varphi[\sigma(x|a)]$ because $\mathcal{N} \vDash \varphi[\sigma(x|0)]$ and $0 \in |\mathcal{N}|$. Thus it must be that $\mathcal{N} \vDash \varphi^x_{\underline{S}x}[\sigma(x|a)]$. By the Substitution Lemma, this means that $\mathcal{N} \vDash \varphi[\sigma(x|\overline{\sigma(x|a)}(\underline{S}x))]$. This means $\mathcal{N} \vDash \varphi[\sigma(x|\underline{S}^{\mathcal{N}}(a))]$. So we have that for any $a \in |\mathcal{N}|$ $(a \in \mathbb{N})$, $\mathcal{N} \vDash \varphi[\sigma(x|(a+1))]$. In our metalanguage, let us refer to $a+1$ as d. Notice that this means that for any $d \in \mathbb{N} \setminus \{0\}$ (because "$a+1$" prevents $d = 0$), $\mathcal{N} \vDash \varphi[\sigma(x|d)]$. However, recall that we also have from our assumption that $\mathcal{N} \vDash \varphi[\sigma(x|0)]$. So, in fact, for any $d \in \mathbb{N}$, $\mathcal{N} \vDash \varphi[\sigma(x|d)]$. Thus our assumption tells us that for any $d \in |\mathcal{N}|$, $\mathcal{N} \vDash \varphi[\sigma(x|d)]$.

We wish to show that $\mathcal{N} \vDash \forall x\varphi[\sigma]$. This is true if and only if for any $d \in |\mathcal{N}|$, $\mathcal{N} \vDash \varphi[\sigma(x|d)]$. Notice that this is precisely what we have from our previous assumption. Thus $\mathcal{N} \vDash \forall x\varphi[\sigma]$. By our line of reasoning, $\mathcal{N} \vDash (\varphi^x_{\underline{0}} \wedge \forall x(\varphi \to \varphi^x_{\underline{S}x})) \to \forall x\varphi$.

Therefore $\mathcal{N}$ models (PA1), (PA2), (PA3), and (PA4). $\square$

(b) Let PA be the set of all L_{Ar}-sentences of the form (PA1)-(PA4). Show that for any $n \in \mathbb{N}$

$$PA \vdash \forall x(\neg(\underbrace{\underline{S}\underline{S}...\underline{S}}_{n+1 \text{ times}}\ x)\dot{=}x)$$

where the term

$$\underbrace{\underline{S}\underline{S}...\underline{S}}_{n+1 \text{ times}}\ x$$

is the variable x preceded by $n+1$ occurrences of the symbol $\underline{S}$.

Proof. We know that $PA \vdash \forall x(\neg \underline{S}x\dot{=}\underline{0})$, because this is of the form (PA1). We also know that $PA \vdash \forall x(\neg \underline{S}x\dot{=}\underline{0}) \rightarrow (\neg \underline{S}x\dot{=}\underline{0})^x_{\underline{S}x}$, since this is of the form (Ax2). By modus ponens, $PA \vdash (\neg \underline{S}x\dot{=}\underline{0})^x_{\underline{S}x}$. Which means that $PA \vdash \neg \underline{S}\underline{S}x\dot{=}\underline{0}$. Since the variable x does not appear free in (PA1)-(PA4), we can use the Generalization Theorem to say that $PA \vdash \forall x(\neg \underline{S}\underline{S}x\dot{=}\underline{0})$.

We can repeat these steps with the new theorem $\forall x(\neg \underline{S}\underline{S}x\dot{=}\underline{0})$, in an inductive argument on the amount of $\underline{S}$ symbols. We come to the conclusion that for any $n \in \mathbb{N}$, $PA \vdash \forall x(\neg(\underbrace{\underline{S}\underline{S}...\underline{S}}_{n+1 \text{ times}}\ x)\dot{=}\underline{0})$.

This means, in particular, that $PA \vdash (\neg(\underbrace{\underline{S}\underline{S}...\underline{S}}_{n+1 \text{ times}}\ \underline{0})\dot{=}\underline{0})$. Let us consider $\varphi = (\neg(\underbrace{\underline{S}\underline{S}...\underline{S}}_{n+1 \text{ times}}\ x)\dot{=}x)$. This means that $PA \vdash \varphi^x_{\underline{0}}$. We would like to show that $PA \vdash \varphi \rightarrow \varphi^x_{\underline{S}x}$.

Consider $PA \vDash \varphi \rightarrow \varphi^x_{\underline{S}x}$. This is true if and only if $PA \vDash \varphi \rightarrow \varphi^x_{\underline{S}x}[\sigma]$, where σ is an arbitrary $\mathcal{N}$-assignment. This is true if and only if $PA \nvDash \varphi[\sigma]$ or $PA \vDash \varphi^x_{\underline{S}x}[\sigma]$. We are done if $PA \nvDash \varphi[\sigma]$, so let us assume that $PA \vDash \varphi[\sigma]$.

Our assumption means that $\langle \underbrace{\underline{S}^{\mathcal{N}}\underline{S}^{\mathcal{N}}...\underline{S}^{\mathcal{N}}}_{n+1 \text{ times}}\overline{\sigma}(x), \overline{\sigma}(x)\rangle \notin \dot{=}^{\mathcal{N}}$. In our metalanguage, our assumption tells us that $x + (n+1) \neq x$.

We wish to show that $PA \vDash \varphi^x_{\underline{S}x}[\sigma]$. By the Substitution Lemma, this is true if and only if $PA \vDash \varphi[\sigma(x|\overline{\sigma}(\underline{S}x))]$. This means $PA \vDash \varphi[\sigma(x|x+1)]$. This means that $\langle \underbrace{\underline{S}^{\mathcal{N}}\underline{S}^{\mathcal{N}}...\underline{S}^{\mathcal{N}}}_{n+1 \text{ times}}\overline{\sigma}(x+1), \overline{\sigma}(x+1)\rangle \notin \dot{=}^{\mathcal{N}}$.

In our metalanguage this means that $(x+1)+(n+1) \neq (x+1)$. Suppose, for the sake of contradiction that, in fact, $(x+1)+(n+1) = (x+1)$. Simple algebra tells us that this is equivalent to $x+(n+1) = x$. However, recall that our assumption tells us that $x+(n+1) \neq x$. So our most recent supposition that $(x+1)+(n+1) = (x+1)$ leads to a contradiction and must be false. Thus it is true that $(x+1)+(n+1) \neq (x+1)$.

So it is true that $\langle \underbrace{\underline{S}^{\mathcal{N}}\underline{S}^{\mathcal{N}}...\underline{S}^{\mathcal{N}}}_{n+1 \text{ times}} \overline{\sigma}(x+1), \overline{\sigma}(x+1)\rangle \notin \doteq^{\mathcal{N}}$, and hence $PA \vDash \varphi^x_{\underline{S}x}[\sigma]$. This means that $PA \vDash \varphi \to \varphi^x_{\underline{S}x}[\sigma]$. Since σ was arbitrary, this means that $PA \vDash \varphi \to \varphi^x_{\underline{S}x}$. By Completeness, it follows that $PA \vdash \varphi \to \varphi^x_{\underline{S}x}$.

Since x does not appear free in PA (because it does not appear free in (PA1)-(PA4)), we can use the Generalization Theorem to see that $PA \vdash \forall x(\varphi \to \varphi^x_{\underline{S}x})$. Recall that we have $PA \vdash \varphi^x_{\underline{0}}$. Thus we have $PA \vdash (\varphi^x_{\underline{0}} \wedge \forall x(\varphi \to \varphi^x_{\underline{S}x}))$. Recall that $PA \vdash$ (PA4). That is, for any $\varphi \in F_{L_{Ar}}$ with $FV(\varphi) = \{x\}$, $PA \vdash (\varphi^x_{\underline{0}} \wedge \forall x(\varphi \to \varphi^x_{\underline{S}x})) \to \forall x\varphi$. Our specific φ only has x as a free variable, so this holds for our formula. Thus by modus ponens on our two most recent results, $PA \vdash \forall x\varphi$.

Therefore,

$$PA \vdash \forall x(\neg(\underbrace{\underline{S}\underline{S}...\underline{S}}_{n+1 \text{ times}}\ x) \doteq x)$$

$\square$

(c) Show that (PA1)-(PA3) are independent:

i. $\{\psi_1\} \nvdash \psi_2$ and $\{\psi_1\} \nvdash \neg\psi_2$

Proof. Suppose for the sake of contradiction that in fact $\{\psi_1\} \vdash \psi_2$. By the Deduction Theorem, it follow that $\vdash \psi_1 \to \psi_2$. By Soundness, this means that $\psi_1 \to \psi_2$ is valid. In order to find a contradiction, we would like to present an L-structure which satisfies the antecedent, but not the consequent. Consider $\mathfrak{A}$ with the properties $|\mathfrak{A}| = \mathbb{N}$, $\underline{0}^{\mathfrak{A}} = 0$, and

$$\underline{S}^{\mathfrak{A}}(n) = \begin{cases} 1, & \text{n odd} \\ 2, & \text{n even} \end{cases}$$

This L-structure leads to a contradiction because it satisfies ψ_1 but does not satisfy ψ_2. So $\psi_1 \to \psi_2$ is not a valid. This is a contradiction, so it must instead be the case that $\{\psi_1\} \nvdash \psi_2$.

Suppose for the sake of contradiction, that $\{\psi_1\} \vdash \neg\psi_2$. By the same reasoning, this supposition leads to the implication that $\psi_1 \to \neg\psi_2$ is valid. In order to find a contradiction, we would like to present an L-structure which satisfies the antecedent, but does not the satisfy consequent. In this case, that is actually equivalent to satisfying both ψ_1 and ψ_2. In fact, we already have such an L-structure. As shown in (a), $\mathcal{N}$ satisfies both ψ_1 and ψ_2. So it must be that $\psi_1 \to \neg\psi_2$ is not valid. This is a contradiction, so it must instead bybe the case that $\{\psi_1\} \nvdash \neg\psi_2$.

Therefore $\{\psi_1\} \nvdash \psi_2$ and $\{\psi_1\} \nvdash \neg\psi_2$. □

ii. $\{\psi_1\} \nvdash \psi_3$ and $\{\psi_1\} \nvdash \neg\psi_3$

Proof. Using the same technique, we would like to show that the implication that $\psi_1 \to \psi_3$ is valid, leads to a contradiction. Consider the L-structure $\mathfrak{A}$ with the properties $\mathfrak{A} = \mathbb{N} \setminus \{0\}$, $\underline{0}^{\mathfrak{A}} = \underline{0}^{\mathcal{N}}$, and $\underline{S}^{\mathfrak{A}}(n) = \underline{S}^{\mathcal{N}}(n)$. This leads to a contradiction because $\mathfrak{A}$ satisfies ψ_1 but it does not satisfy ψ_3, and hence $\psi_1 \to \psi_3$ is not valid. So it must instead be the case that $\{\psi_1\} \nvdash \psi_3$.

Suppose for the sake of contradiction, that $\{\psi_1\} \vdash \neg\psi_3$. By the same reasoning, this supposition leads to the implication that $\psi_1 \to \neg\psi_3$ is valid. In order to find a contradiction, we would like to present an L-structure which satisfies the antecedent, but does not satisfy the consequent. In this case, that is actually equivalent to satisfying both ψ_1 and ψ_3. In fact, we already have such an L-structure. As shown in (a), $\mathcal{N}$ satisfies both ψ_1 and ψ_3. So it must be that $\psi_1 \to \neg\psi_3$ is not valid. This is a contradiction, so it must instead be the case that $\{\psi_1\} \nvdash \neg\psi_3$.

Therefore $\{\psi_1\} \nvdash \psi_3$ and $\{\psi_1\} \nvdash \neg\psi_3$. □

iii. $\{\psi_2\} \nvdash \psi_1$ and $\{\psi_2\} \nvdash \neg\psi_1$

Proof. Using the same technique, we would like to show that the implication that $\psi_2 \to \psi_1$ is valid leads to a contradiction. Consider the L-structure $\mathfrak{A}$ with the properties $|\mathfrak{A}| = \{0\}$, $\underline{0}^{\mathfrak{A}} = 0$, and $\underline{S}^{\mathfrak{A}}(n) = n$. $\mathfrak{A}$ satisfies ψ_2 but it does not satisfy ψ_1. This means that $\psi_2 \to \psi_1$ is not valid, but that is a contradiction. So it must instead be the case that $\{\psi_2\} \nvdash \psi_1$.

Suppose, for the sake of contradiction that $\{\psi_2\} \vdash \neg\psi_1$. By Contraposition, this is true if and only if $\{\psi_1\} \vdash \neg\psi_2$. However, we showed in i that in fact $\{\psi_1\} \nvdash \neg\psi_2$. So it must instead be the case that $\{\psi_2\} \nvdash \neg\psi_1$.

Therefore $\{\psi_2\} \nvdash \psi_1$ and $\{\psi_2\} \nvdash \neg\psi_1$. $\square$

iv. $\{\psi_2\} \nvdash \psi_3$ and $\{\psi_2\} \nvdash \neg\psi_3$

Proof. Using the same technique, we would like to show that the implication that $\psi_2 \to \psi_3$ is valid leads to a contradiction. Consider the L-structure $\mathfrak{A}$ with the properties $|\mathfrak{A}| = \mathbb{N}\backslash\{0\}$, $\underline{0}^{\mathfrak{A}} = \underline{0}^{\mathcal{N}}$, and $\underline{S}^{\mathfrak{A}}(n) = \underline{S}^{\mathcal{N}}(n)$. $\mathfrak{A}$ satisfies ψ_2, but it does not satisfy ψ_3, so this means that $\psi_2 \to \psi_3$ is not valid. This is a contradiction, so it must instead be the case that $\{\psi_2\} \nvdash \psi_3$.

Suppose for the sake of contradiction, that $\{\psi_2\} \vdash \neg\psi_3$. By the same reasoning, this supposition leads to the implication that $\psi_2 \to \neg\psi_3$ is valid. In order to find a contradiction, we would like to present an L-structure which satisfies the antecedent, but does not the satisfy consequent. In this case, that is actually equivalent to satisfying both ψ_2 and ψ_3. In fact, we already have such an L-structure. As shown in (a), $\mathcal{N}$ satisfies both ψ_2 and ψ_3. So it must be that $\psi_2 \to \neg\psi_3$ is not valid. This is a contradiction, so it must instead be the case that $\{\psi_2\} \nvdash \neg\psi_3$.

Therefore $\{\psi_2\} \nvdash \psi_3$ and $\{\psi_2\} \nvdash \neg\psi_3$. $\square$

v. $\{\psi_3\} \nvdash \psi_1$ and $\{\psi_3\} \nvdash \neg\psi_1$

Proof. Using the same technique, we would like to show that the implication that $\psi_3 \to \psi_1$ is valid leads to a contradiction. Consider the L-structure $\mathfrak{A}$ with the properties $|\mathfrak{A}| = \{0\}$, $\underline{0}^{\mathfrak{A}} = 0$, and $\underline{S}^{\mathfrak{A}}(n) = n$. $\mathfrak{A}$ satisfies ψ_3, but it does not satisfy ψ_1. This means that $\psi_3 \to \psi_1$ is not valid, but that is a contradiction. So it must instead be the case that $\{\psi_3\} \nvdash \psi_1$.

Suppose, for the sake of contradiction that $\{\psi_3\} \vdash \neg\psi_1$. By Contraposition, this is true if and only if $\{\psi_1\} \vdash \neg\psi_3$. However, we showed in ii that in fact $\{\psi_1\} \nvdash \neg\psi_3$. So it must instead be the case that $\{\psi_3\} \nvdash \neg\psi_1$.

Therefore $\{\psi_3\} \nvdash \psi_1$ and $\{\psi_3\} \nvdash \neg\psi_1$. $\square$

vi. $\{\psi_3\} \nvdash \psi_2$ and $\{\psi_3\} \nvdash \neg\psi_2$

Proof. Using the same technique, we would like to show that the implication that $\psi_3 \to \psi_2$ is valid leads to a contradiction. Consider the L-structure $\mathfrak{A}$ with the properties $|\mathfrak{A}| = \{0, 1, 2\}$, $\underline{0}^{\mathfrak{A}} = 0$, and

$$\underline{S}^{\mathfrak{A}}(n) = \begin{cases} 1, & \text{n odd} \\ 2, & \text{n even} \end{cases}$$

$\mathfrak{A}$ satisfies ψ_3 but it does not satisfy ψ_2. So it follows that $\psi_3 \to \psi_2$ is not valid. This is a contradiction. So it must instead be the case that $\{\psi_3\} \nvdash \psi_2$.

Suppose, for the sake of contradiction that $\{\psi_3\} \vdash \neg\psi_2$. By Contraposition, this is true if and only if $\{\psi_2\} \vdash \neg\psi_3$. However, we showed in iv that in fact $\{\psi_2\} \nvdash \neg\psi_3$. So it must instead be the case that $\{\psi_3\} \nvdash \neg\psi_2$.

Therefore $\{\psi_3\} \nvdash \psi_2$ and $\{\psi_3\} \nvdash \neg\psi_2$. □

Problem 2. A collection $\mathcal{A}$ of L-structure is said to be *finitely axiomatizable* if there is a finite set of L-formulas $\Gamma \subseteq F_L$ such that $\mathfrak{A}$ satisfies Γ iff $\mathfrak{A} \in \mathcal{A}$. In this case, we say that Γ *axiomatizes* $\mathcal{A}$.

Consider a language L with no constant or function symbols and whose only predicate symbol is the two-place predicate $\dot{<}$. (Assume L contains the equality predicate $\doteq$).

(a) Recall that a relation is called a *linear ordering* just in case it is reflexive, anti-symmetric, transitive, and connected. Show that the collection of L-structure $\mathfrak{A}$ such that $\dot{<}^{\mathfrak{A}}$ is a linear ordering is finitely axiomatizable.

Proof. Let $\mathcal{A}$ be the collection of L-structures $\mathfrak{A}$ such that $\dot{<}^{\mathfrak{A}}$ is a linear ordering. Let $\Gamma = \{\psi_1, \psi_2, \psi_3, \psi_4\}$, where

$$\begin{aligned}
\psi_1 &:= \forall x(x \dot{<} x) \\
\psi_2 &:= \forall x \forall y(x \dot{<} y \wedge y \dot{<} x \to x \doteq y) \\
\psi_3 &:= \forall x \forall y \forall z(x \dot{<} y \wedge y \dot{<} z \to x \dot{<} z) \\
\psi_4 &:= \forall x \forall y(\neg x \doteq y \to (x \dot{<} y \vee y \dot{<} x))
\end{aligned}$$

Suppose that some L-structure $\mathfrak{A}$ satisfies Γ. That is to say $\mathfrak{A}$ satisfies ψ_1, ψ_2, ψ_3, and ψ_4. In our metalanguage, we can see that:

- $\mathfrak{A}$ satisfies ψ_1 if and only if $\dot{<}^{\mathfrak{A}}$ is *reflexive*
- $\mathfrak{A}$ satisfies ψ_2 if and only if $\dot{<}^{\mathfrak{A}}$ is *anti-symmetric*
- $\mathfrak{A}$ satisfies ψ_3 if and only if $\dot{<}^{\mathfrak{A}}$ is *transitive*
- $\mathfrak{A}$ satisfies ψ_4 if and only if $\dot{<}^{\mathfrak{A}}$ is *connected*

This means that $\mathfrak{A}$ satisfies Γ if and only if $\dot{<}^{\mathfrak{A}}$ is *reflexive*, *anti-symmetric*, *transitive*, and *connected*. This is precisely the definition of $\dot{<}^{\mathfrak{A}}$ being a linear ordering. We have shown that if some $\mathfrak{A}$ satisfies Γ, then it must be the case that $\dot{<}^{\mathfrak{A}}$ is a linear ordering. Thus $\mathfrak{A} \in \mathcal{A}$.

Suppose that $\mathfrak{A} \in \mathcal{A}$. Then we have $\dot{<}^{\mathfrak{A}}$ is a linear ordering. This means that $\dot{<}^{\mathfrak{A}}$ is *reflexive*, *anti-symmetric*, *transitive*, and *connected*. From earlier, we see that $\mathfrak{A}$ satisfies ψ_1 because $\dot{<}^{\mathfrak{A}}$ is *reflexive*, $\mathfrak{A}$ satisfies ψ_2 because $\dot{<}^{\mathfrak{A}}$ is *anti-symmetric*, $\mathfrak{A}$ satisfies ψ_3 because $\dot{<}^{\mathfrak{A}}$ is *transitive*, and $\mathfrak{A}$ satisfies ψ_4 because $\dot{<}^{\mathfrak{A}}$ is *connected*. This means that $\mathfrak{A}$ satisfies Γ. We have shown that if $\mathfrak{A} \in \mathfrak{A}$, then $\mathfrak{A}$ satisfies Γ.

$\mathfrak{A}$ satisfies Γ if and only if $\mathfrak{A} \in \mathfrak{A}$. Since Γ is a finite set (with 4 elements), our definition of "finitely axiomatizable" is satsified. Therefore, the collection of L-structures $\mathfrak{A}$ such that $\dot{<}^{\mathfrak{A}}$ is a linear ordering (which we referred as $\mathcal{A}$) is finitely axiomatizable. $\square$

(b) A two-place relation R on a set A is called a *well-ordering* if it is a linear ordering and every non-empty subset of A has an R-least element, i.e., for every $B \subseteq A$, if $B \neq \varnothing$, then there is an $a \in B$ such that for all $b \in B$, $\langle a, b\rangle \in R$.

Let $\mathcal{A}_{wo}$ be the collection of L-structures such that $\dot{<}^{\mathfrak{A}}$ is a well-ordering. In the following, you will show that $\mathcal{A}_{wo}$ is not finitely axiomatizable.

Suppose that $\Gamma \subseteq F_L$ is a finite set of L-formulas axiomatizing $\mathfrak{A}_{wo}$. Consider a language L' containing $\dot{<}$ and countably many new constant symbols $c_0, c_1, c_2, \ldots.$ For each $j \in \mathbb{N}$ define and L'-formula

$$\varphi_j = (c_{j+1} \dot{<} c_j) \wedge (\neg c_{j+1} \doteq c_j)$$

i. Show that for any $j \in \mathbb{N}$, the set

$$\Gamma \cup \{\varphi_j\}$$

is satisfiable.

Proof. Consider some arbitrary $\mathfrak{A} \in \mathcal{A}_{wo}$. Let $\mathfrak{A}'$ be the same L-structure with the modification that each constant of L' is assigned such that $c_j^{\mathfrak{A}'} > c_{j+1}^{\mathfrak{A}'}$ for every $j \in \mathbb{N}$. Since these constants do not appear in L, $\mathfrak{A}'$ satisfies Γ (and hence each formula of Γ). By the way that we have assigned constants in $\mathfrak{A}'$, φ_j is also satisfied by $\mathfrak{A}'$ for any $j \in \mathbb{N}$. So for any $j \in \mathbb{N}$, $\mathfrak{A}'$ satisfies $\Gamma \cup \{\varphi_j\}$. Therefore, for any $j \in \mathbb{N}$, $\Gamma \cup \{\varphi_j\}$ is satisfiable. □

ii. Show that for any $k \in \mathbb{N}$, the set

$$\Gamma \cup \{\varphi_j | j \leq k\}$$

is satisfiable.

Proof. Consider $\mathfrak{A}'$ from i.. As determined in i. we know that $\mathfrak{A}'$ satisfies Γ and $\mathfrak{A}'$ satisfies φ_j for any $j \in \mathbb{N}$. This means for any $k \in \mathbb{N}$, $\mathfrak{A}'$ satisfies $\{\varphi_j | j \leq k\}$, because all the indices such that $j \leq k$ are in $\mathbb{N}$. It follows that $\mathfrak{A}'$ satisfies $\Gamma \cup \{\varphi_j | j \leq k\}$. Therefore $\Gamma \cup \{\varphi_j | j \leq k\}$ is satisfiable. □

iii. Show that the set

$$\Gamma' := \Gamma \cup \{\varphi_j | j \in \mathbb{N}\}$$

is satisfiable.

Proof. In i. and ii., we showed that every finite subset of Γ' is satisfiable. Therefore, by the Compactness Theorem, Γ' is satisfiable. □

iv. Show that if $\mathfrak{B}$ is an L'-structure satisfying Γ' then $\dot{<}^{\mathfrak{B}}$ is not a well-ordering.

Proof. Suppose, for the sake of contradiction that $\dot{<}^{\mathfrak{B}}$ is a well-ordering. In particular, suppose that there is some $c_m^{\mathfrak{B}}$ of every nonempty subset of $|\mathfrak{B}|$ such that $c_m^{\mathfrak{B}}$ is the least $\dot{<}^{\mathfrak{B}}$-least element of the subset. Since $\mathfrak{B}$ satisfies Γ', it must be also be the case that $\mathfrak{B}$ satisfies φ_m. That is to say, $c_{m+1}^{\mathfrak{B}} \dot{<}^{\mathfrak{B}} c_m^{\mathfrak{B}}$ and $c_{m+1}^{\mathfrak{B}} \neq c_m^{\mathfrak{B}}$. This means $c_m^{\mathfrak{B}}$ is not actually the $\dot{<}^{\mathfrak{B}}$-least element of the subset of $|\mathfrak{B}|$. This is a contradiction, so no such element can exist in this context. Therefore, if $\mathfrak{B}$ is an L'-structure satisfying Γ' then $\dot{<}^{\mathfrak{B}}$ cannot be a well-ordering. $\square$

v. Show there is an L-structure $\mathfrak{A}$ satisfying Γ such that $\mathfrak{A} \notin \mathcal{A}_{wo}$. Conclude that $\mathcal{A}_{wo}$ is not finitely axiomatizable.

Proof. From iii we know that Γ' is satisfiable. Let $\mathfrak{A}$ be an L'-stcuture which satisfies Γ'. By the way we defined L' in relation to L, $\mathfrak{A}$ is also an L-structure. Since $\Gamma \subseteq \Gamma'$, $\mathfrak{A}$ satisfies Γ as well. Recall that we supposed that Γ is a finite axiomatization of $\mathcal{A}_{wo}$. Since $\mathfrak{A}$ is an L-structure which satisfies Γ, it follows that $\mathfrak{A} \in \mathcal{A}_{wo}$. However, we showed in iv that if $\mathfrak{A}$ is an L'-structure satisfying Γ', then $\dot{<}^{\mathfrak{A}}$ is not a well-ordering. Since $\dot{<}^{\mathfrak{A}}$ is not a well-ordering, $\mathfrak{A} \notin \mathcal{A}_{wo}$. This is a contradiction, and therefore our first supposition that exists such a finite set Γ which axiomatizes $\mathcal{A}_{wo}$ must be incorrect. Therefore, $\mathcal{A}_{wo}$ is not finitely axiomatizable. $\square$

(c) Show that the collection of L-structures $\mathfrak{A}$ such that $|\mathfrak{A}|$ is uncountable is not finitely axiomatizable.

Proof. Let $\mathcal{A}$ be the set of L-structures $\mathfrak{A}$ such that $|\mathfrak{A}|$ is uncountable. Suppose there exists a set Γ which axiomatizes $\mathcal{A}$. By definition of axiomatization, this means that for any $\mathfrak{A} \in \mathcal{A}$, $\mathfrak{A}$ satisfies Γ. Let $\hat{\mathfrak{A}}$ be an arbitrary L-structure in $\mathcal{A}$. We have that $\hat{\mathfrak{A}}$ satisfies Γ. By the Downhill Lowenheim-Skolem Theorem, since Γ is satisfiable, there exists some L-structure $\mathfrak{B}$ which also satisfies Γ and whose domain is countable or finite. Since this $\mathfrak{B}$ satisfies Γ, and since Γ axiomatizes $\mathcal{A}$, it follows that $\mathfrak{B} \in \mathcal{A}$. But, since $\mathfrak{B}$ has a countable or finite domain, in cannot be the case that $\mathfrak{B} \in \mathcal{A}$. This is because $\mathcal{A}$ is the set of L-structures whose domains are uncountable. So $\mathfrak{B} \notin \mathcal{A}$ because a domain cannot be both countable (or finite) and uncountable. We have arrive at the contradiction that $\mathfrak{B}$ both is an is not an element of $\mathcal{A}$. Hence our assumption that there exists a set Γ which axiomatizes $\mathcal{A}$ is false.

So there is no set Γ which axiomatizes $\mathcal{A}$. In particular, there is no finite set Γ which axiomatizes $\mathcal{A}$. This means that $\mathcal{A}$ is not finitely axiomatization. Therefore, the collection of L-structures $\mathfrak{A}$ such that $|\mathfrak{A}|$ is uncountable is not finitely axiomatizable. $\square$

Problem 3. Recall the language L_{ST} of set theory with symbols:

- Constant Symbols: none
- Predicate Symbols: the two-place predicate $\underline{\in}$ (and equality $\doteq$)
- Function Symbols: none

Recall the axioms of pair and union:

$$\varphi_{pair} := \forall x \forall y \exists z \forall w (w \underline{\in} z \equiv (w \doteq x \vee w \doteq y))$$
$$\varphi_{union} := \forall x \exists y \forall z (z \underline{\in} y \equiv \exists w (w \underline{\in} x \wedge z \underline{\in} w))$$

(a) Consider the L_{ST}-structure $\mathfrak{A}$ with $|\mathfrak{A}| = \mathbb{N}$ and

$$\underline{\in}^{\mathfrak{A}} = \ < \ := \{\langle n, m \rangle \in \mathbb{N} \times \mathbb{N} | n < m\}$$

Show that $\mathfrak{A} \vDash \varphi_{union}$ but $\mathfrak{A} \nvDash \varphi_{pair}$.

Proof. Let us first consider $\mathfrak{A} \vDash \varphi_{union}$. This is true if and only if $\mathfrak{A} \vDash \forall x \exists y \forall z (z \underline{\in} y \equiv \exists w (w \underline{\in} x \wedge z \underline{\in} w))[\sigma]$, where σ is an arbitrary $\mathfrak{A}$-assignment. This is true if and only if for any $a \in |\mathfrak{A}|$, there is some $b \in |\mathfrak{A}|$ such that for any $c \in |\mathfrak{A}|$,

$$\mathfrak{A} \vDash (z \underline{\in} y \equiv \exists w (w \underline{\in} x \wedge z \underline{\in} w))[\sigma(x|a)(y|b)(z|c)].$$

This is true if and only if $\mathfrak{A} \vDash z \underline{\in} y[\sigma(x|a)(y|b)(z|c)]$ and $\mathfrak{A} \vDash \exists w (w \underline{\in} x \wedge z \underline{\in} w)[\sigma(x|a)(y|b)(z|c)]$, or $\mathfrak{A} \nvDash z \underline{\in} y[\sigma(x|a)(y|b)(z|c)]$ and $\mathfrak{A} \nvDash \exists w (w \underline{\in} x \wedge z \underline{\in} w)[\sigma(x|a)(y|b)(z|c)]$.

Let us check if $\mathfrak{A} \vDash z \underline{\in} y[\sigma(x|a)(y|b)(z|c)]$. This is true if and only if $\langle c, b \rangle \in \underline{\in}^{\mathfrak{A}}$. In our metalanguage, this means that there is some $b \in \mathbb{N}$ such that for any $c \in \mathbb{N}$, $c < b$. We know this to be false, as there is no greatest natural number (or greatest integer). For if there were such a natural number M, $M + 1$ would be greater than M. This is a contradiction since we assumed M to be greater than every natural number. Thus it follows that $\mathfrak{A} \nvDash z \underline{\in} y[\sigma(x|a)(y|b)(z|c)]$.

Let us check if $\mathfrak{A} \vDash \exists w (w \underline{\in} x \wedge z \underline{\in} w)[\sigma(x|a)(y|b)(z|c)]$. This is true if and only if for any $a \in |\mathfrak{A}|$, there is some $b \in |\mathfrak{A}|$ such that

for any $c \in |\mathfrak{A}|$, there exists some $d \in |\mathfrak{A}|$ such that $\mathfrak{A} \vDash (w \underline{\in} x \wedge z \underline{\in} w)[\sigma(x|a)(y|b)(z|c)(w|d)]$. This is true if and only if both $\mathfrak{A} \vDash (w \underline{\in} x)[\sigma(x|a)(y|b)(z|c)(w|d)]$ and $\mathfrak{A} \vDash (z \underline{\in} w)[\sigma(x|a)(y|b)(z|c)(w|d)]$.
Let us check if $\mathfrak{A} \vDash (w \underline{\in} x)[\sigma(x|a)(y|b)(z|c)(w|d)]$. This is true if and only if $\langle d, a \rangle \in \underline{\in}^{\mathfrak{A}}$. In our metalanguage, this means that for any $a \in \mathbb{N}$, there exists some $d \in \mathbb{N}$ such that $d < a$. This cannot be the true, as we can see by considering the case $a = 0$. There is no such element $d \in \mathbb{N}$ such that $d < 0$. Thus, in cannot be the case that $\mathfrak{A} \vDash (w \underline{\in} x)[\sigma(x|a)(y|b)(z|c)(w|d)]$.
It follows that it cannot be the case that both

$$\mathfrak{A} \vDash (w \underline{\in} x)[\sigma(x|a)(y|b)(z|c)(w|d)] \text{ and } \mathfrak{A} \vDash (z \underline{\in} w)[\sigma(x|a)(y|b)(z|c)(w|d)].$$

Hence, $\mathfrak{A} \nvDash \exists w(w \underline{\in} x \wedge z \underline{\in} w)[\sigma(x|a)(y|b)(z|c)]$.
We have shown that both $\mathfrak{A} \nvDash z \underline{\in} y[\sigma(x|a)(y|b)(z|c)]$ and $\mathfrak{A} \nvDash \exists w(w \underline{\in} x \wedge z \underline{\in} w)[\sigma(x|a)(y|b)(z|c)]$. Thus, $\mathfrak{A} \vDash \varphi_{union}$.

Let us now consider $\mathfrak{A} \vDash \varphi_{pair}$. This is true if and only if $\mathfrak{A} \vDash \forall x \forall y \exists z \forall w(w \underline{\in} z \equiv (w \dot{=} x \vee w \dot{=} y))[\sigma]$, where σ is an arbitrary $\mathfrak{A}$-assignment. This is true if for any $a \in \mathbb{N}$, for any $b \in \mathbb{N}$, there is some $c \in \mathbb{N}$ such that for any $d \in \mathbb{N}$, $\mathfrak{A} \vDash (w \underline{\in} z \equiv (w \dot{=} x \vee w \dot{=} y))[\sigma(x|a)(y|b)(z|c)(w|d)]$. In our metalanguage, this means that for any $a \in \mathbb{N}$, for any $b \in \mathbb{N}$, there is some $c \in \mathbb{N}$ such that for any $d \in \mathbb{N}$, $d < c$ implies that $d = d$ or $d = b$, and $d = a$ or $d = b$ implies $d < c$.
Consider the case that $a = 9$, $b = 10$. There is no such c which would restrict d to either be 9 or 10. Since we have have the assumption that $d < c$, c must be 11 or greater in order for the possibility of $d = 9$ or $d = 10$. However, if $c > 11$ and $d < c$, it is not necessarily the case that $d = 9$ or $d = 10$, because d could also be any natural from 1 to 8. Thus, by means of counterexample, we have shown that under these conditions, $\mathfrak{A} \nvDash (w \underline{\in} z \equiv (w \dot{=} x \vee w \dot{=} y))[\sigma(x|a)(y|b)(z|c)(w|d)]$. This means that $\mathfrak{A} \nvDash \forall x \forall y \exists z \forall w(w \underline{\in} z \equiv (w \dot{=} x \vee w \dot{=} y))[\sigma]$. Thus, $\mathfrak{A} \nvDash \varphi_{pair}$.
Therefore we have shown $\mathfrak{A} \vDash \varphi_{union}$ but $\mathfrak{A} \nvDash \varphi_{pair}$. $\square$

(b) Define a sequence of sets recursively as follows:

$$V_0 := \varnothing$$
$$v_{n+1} := \mathcal{P}(V_n)$$

Define the set V_ω of *hereditarily finite sets* by

$$V_\omega = \bigcup_{n \in \mathbb{N}} V_n$$

Consider the L_{ST}-structure $\mathfrak{B}$ with $|\mathfrak{B}| = V_\omega$ and

$$\underline{\in}^{\mathfrak{B}} = \in := \{\langle x, y \rangle \in V_\omega \times V_\omega | x \in y\}$$

Show that $\mathfrak{B} \vDash \varphi_{union}$ and $\mathfrak{B} \vDash \varphi_{pair}$.

Proof. Let us first consider $\mathfrak{B} \vDash \varphi_{union}$. By a similar line of reasoning as part (a), this is true if and only if for and $a \in V_\omega$, there is a $b \in V_\omega$ such that for any $c \in V_\omega$, $\mathfrak{B} \vDash z \underline{\in} y[\sigma(x|a)(y|b)(z|c)]$ and $\mathfrak{B} \vDash \exists w(w \underline{\in} x \wedge z \underline{\in} w)[\sigma(x|a)(y|b)(z|c)]$, or $\mathfrak{B} \nvDash z \underline{\in} y[\sigma(x|a)(y|b)(z|c)]$ and $\mathfrak{B} \nvDash \exists w(w \underline{\in} x \wedge z \underline{\in} w)[\sigma(x|a)(y|b)(z|c)]$.

Let us check $\mathfrak{B} \vDash z \underline{\in} y[\sigma(x|a)(y|b)(z|c)]$. This is true if and only if, as interpreted in our metalanguage, there is some $b \in V_\omega$ such that for any $c \in V_\omega$, $c \in b$. Suppose that this is true. Since $b \in V_\omega$, b is an element of some V_n. This means that $\{b\} \in V_{n+1}$, and hence $\{b\}$ is also an element of V_ω. However $\{b\} \notin b$, and this contradicts our supposition. Therefore no such element b can exists, and hence $\mathfrak{B} \nvDash z \underline{\in} y[\sigma(x|a)(y|b)(z|c)]$.

Let us check $\mathfrak{B} \vDash \exists w(w \underline{\in} x \wedge z \underline{\in} w)[\sigma(x|a)(y|b)(z|c)]$. This is true if and only if for all $a \in V_\omega$, there is some $b \in V_\omega$ such that for any $c \in V_\omega$, there is some $d \in V_\omega$ such that $\mathfrak{B} \vDash (w \underline{\in} x \wedge z \underline{\in} w)[\sigma(x|a)(y|b)(z|c)(w|d)]$. In our metalanguage this is interpreted to mean for all $a \in V_\omega$, there is some $b \in V_\omega$ such that for any $c \in V_\omega$, there is some $d \in V_\omega$ such that $d \in a$ and $c \in d$. Consider the case that $a = \varnothing$. It cannot be the case that there is some $d \in V_\omega$ such that $d \in a$, because in the case a has no elements.

Thus it cannot be the case that $\mathfrak{B} \vDash \exists w(w \underline{\in} x \wedge z \underline{\in} w)[\sigma(x|a)(y|b)(z|c)]$. So $\mathfrak{B} \nvDash \exists w(w \underline{\in} x \wedge z \underline{\in} w)[\sigma(x|a)(y|b)(z|c)]$. We have shown that $\mathfrak{B} \nvDash z \underline{\in} y[\sigma(x|a)(y|b)(z|c)]$ and $\mathfrak{B} \nvDash \exists w(w \underline{\in} x \wedge z \underline{\in} w)[\sigma(x|a)(y|b)(z|c)]$, and thus $\mathfrak{B} \vDash \varphi_{union}$.

Let us consider $\mathfrak{B} \vDash \varphi_{pair}$. This is true if and only if for any $a \in V_\omega$, for any $b \in V_\omega$, there is some $c \in V_\omega$ such that for any $d \in V_\omega$, $\mathfrak{B} \vDash (w \underline{\in} z \equiv (w \doteq x \vee w \doteq y))[\sigma(x|a)(y|b)(z|c)(w|d)]$, where σ is an arbitrary $\mathfrak{B}$-assignment. This is true if and only if $\mathfrak{B} \vDash$

$(w\dot{\in}z)[\sigma(x|a)(y|b)(z|c)(w|d)]$ and $\mathfrak{B} \models (w\doteq x \vee w\doteq y)[\sigma(x|a)(y|b)(z|c)(w|d)]$, or $\mathfrak{B} \not\models (w\dot{\in}z)[\sigma(x|a)(y|b)(z|c)(w|d)]$ and $\mathfrak{B} \not\models (w\doteq x \vee w\doteq y)[\sigma(x|a)(y|b)(z|c)(w|d)]$.

Let us check $\mathfrak{B} \models (w\dot{\in}z)[\sigma(x|a)(y|b)(z|c)(w|d)]$. This is true if and only if, as interpreted in our metalanguage, there exists some $c \in V_\omega$ such that for any $d \in V_\omega$, $d \in c$. We have previously shown this to be false, earlier in this proof, so $\mathfrak{B} \not\models (w\dot{\in}z)[\sigma(x|a)(y|b)(z|c)(w|d)]$.

Let us check $\mathfrak{B} \models (w\doteq x \vee w\doteq y)[\sigma(x|a)(y|b)(z|c)(w|d)]$. This is true if and only if, $\mathfrak{B} \models (w\doteq x)[\sigma(x|a)(y|b)(z|c)(w|d)]$ or $\mathfrak{B} \models (w\doteq y)[\sigma(x|a)(y|b)(z|c)(w|d)]$. The first of these is true if and only if for any $a \in V_\omega$, for and $b \in V_\omega$, $a = b$. This is false, because $\varnothing \in V_\omega$ and $\{\varnothing\} \in V_\omega$, but $\varnothing \neq \{\varnothing\}$. The second of these is false, for the same exact reason. Thus, $\mathfrak{B} \not\models (w\doteq x \vee w\doteq y)[\sigma(x|a)(y|b)(z|c)(w|d)]$.

We have shown that $\mathfrak{B} \not\models (w\dot{\in}z)[\sigma(x|a)(y|b)(z|c)(w|d)]$ and $\mathfrak{B} \not\models (w\doteq x \vee w\doteq y)[\sigma(x|a)(y|b)(z|c)(w|d)]$. Thus, it follows that $\mathfrak{B} \models \varphi_{pair}$.

Therefore, we have shown that $\mathfrak{B} \models \varphi_{union}$ and $\mathfrak{B} \models \varphi_{pair}$. □

(c) Show that $\{\varphi_{union}\} \not\vdash \varphi_{pair}$ and $\{\varphi_{union}\} \not\vdash \neg\varphi_{pair}$.

Proof. Suppose, for the sake of contradiction that in fact $\{\varphi_{union}\} \vdash \varphi_{pair}$. By the Deduction Theorem, this would imply that $\vdash \varphi_{union} \to \varphi_{pair}$. By Soundness, this implies that $\models \varphi_{union} \to \varphi_{pair}$. This means that for any L-structure $\mathfrak{A}$ and any $\mathfrak{A}$-assignment, $\mathfrak{A} \models \varphi_{union} \to \varphi_{pair}$. In order to find a contradiction, we would like to present an L-structure which satisfies φ_{union} but does not satisfy φ_{pair}. Notice that the L-structure, $\mathfrak{A}$ from (a) does precisely that. Thus, we have arrived at a contradiction and it must be the case instead that $\{\varphi_{union}\} \not\vdash \varphi_{pair}$.

Suppose for the sake of contradiction that in fact $\{\varphi_{union}\} \vdash \neg\varphi_{pair}$. By the same reasoning as before, this leads us to the same implication that $\varphi_{union} \to \neg\varphi_{pair}$ is valid. In order to find a contradiction, we would like to present an L-structure which satisfies φ_{union}, but does not satisfy $\neg\varphi_{pair}$. Equivalently, such an L-structure would have to satisfy φ_{union} and φ_{pair}. Notice that the L-structure $\mathfrak{B}$ from (b) does precisely that. Thus we have arrived at a contradiction and it must be the case instead that $\{\varphi_{union}\} \not\vdash \neg\varphi_{pair}$.

Therefore, we have shown that $\{\varphi_{union}\} \not\vdash \varphi_{pair}$ and $\{\varphi_{union}\} \not\vdash \neg\varphi_{pair}$. □